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Multiple periodic solutions for asymptotically linear Duffing equations with resonance ☆

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ABSTRACT

We investigate multiple periodic solutions of asymptotically linear Duffing equation with resonance using index theory and Morse theory and obtain a new result.

1. Introduction and main results

We consider periodic solutions of the Duffing equation:

$$x''(t) + f\big(t, x(t)\big) = 0, \tag{1.1}$$

$$x(1) - x(0) = 0 = x'(1) - x'(0), \tag{1.2}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous, and meanwhile, we assume that $f' : [0,1] \times \mathbf{R} \to \mathbf{R}$ is also continuous, where f' denotes the derivative with respect to x. Suppose that f satisfies the condition:

(H_1) $$\int_0^1 f(t, +\infty)\, dt > 0 > \int_0^1 f(t, -\infty)\, dt,$$

where $f(t, +\infty) = \liminf_{x \to +\infty} f(t,x)$ and $f(t,-\infty) = \limsup_{x \to -\infty} f(t,x)$.

Our main result is the following theorem.

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Theorem 1.1. *Assume λ and $\lambda_0 \in L^\infty[0,1]$ and $f \in C^1([0,1]\times \mathbf{R}, \mathbf{R})$ satisfies (H_1), and the following two conditions*:

(H_2) $0 \leqslant \frac{f(t,x)}{x} < \lambda(t)$ *as* $|x| \geqslant r > 0$, *r is a constant, with* $i(\lambda)=1$, $\nu(\lambda)=0$.
(H_3) $f(t,0)=0$, $\lambda_0(t)=f'(t,0)$ *and* $1 \bar{\in} [i(\lambda_0), i(\lambda_0)+\nu(\lambda_0)]$.

Then (1.1)–(1.2) *has at least one nontrivial solution.*
Moreover, if we assume

(H_4) $\nu(\lambda_0)=0$, $i(\lambda_0)\geqslant 3$,

then (1.1)–(1.2) *has at least two nontrivial solutions.*

Here for any $a \in L^\infty[0,1]$, $i(a)$ and $\nu(a)$ denote its index and nullity of associated linear Duffing equations (see [1,14] for references). In Section 2, we will briefly recall the index and its properties. For the readers' convenience, we give an example: Assume λ is a constant. Then

$$i(\lambda)=\begin{cases}0 & \text{as } \lambda \leqslant 0,\\ 2k+1 & \text{as } \lambda \in (4k^2\pi^2, 4(k+1)^2\pi^2],\end{cases} \quad \text{and} \quad \nu(\lambda)=\begin{cases}0 & \text{as } \lambda \neq 4k^2\pi^2,\\ 1 & \text{as } \lambda = 0,\\ 2 & \text{as } \lambda = 4k^2\pi^2,\ k\in \mathbf{N}^+.\end{cases}$$

In [1], an index for the second order linear Hamiltonian systems was defined. And in [14], an index for more general linear self-adjoint operator equations was developed. In [5–7,9], by Conley, Zehnder and Long, an index theory for symplectic paths was defined. More applications about this index theory can be found in [11,12,2,3,10,8]. As in [10], throughout this paper, for $a_1, a_2 \in L^\infty[0,1]$, we write $a_1 \leqslant a_2$, if $a_2(t)-a_1(t)\geqslant 0$, for a.e. $t\in[0,1]$; we write $a_1 < a_2$, if $a_1 \leqslant a_2$, and $a_2(t)-a_1(t)>0$ holds on a subset of $[0,1]$ with nonzero measure.

There are many results for (1.1)–(1.2) in literature. It is well known [4] that under conditions

$$(2k\pi)^2+\delta \leqslant \frac{f(t,x)}{x} \leqslant \big(2(k+1)\pi\big)^2-\delta, \quad \text{as } |x|>r>0,\ k\in\mathbf{N},$$

(1.1)–(1.2) has at least one solution. Such conditions are called nonresonant. Resonant conditions in [16,15]:

$$(2k\pi)^2 \leqslant \frac{f(t,x)}{x} \leqslant \big(2(k+1)\pi\big)^2, \quad \text{as } |x|>r>0,\ k\in\mathbf{N},$$

are not enough for existence solutions of (1.1)–(1.2). An additional condition called the (LL) condition like (H_1) is needed. These three papers [4,16,15] are about existence of solutions.

In [13], the author, using the Morse theory, investigated the nontrivial solutions of operator equations $Ax+dg(x)=\theta$ with resonance, where A is a bounded self-adjoint operator defined on a Hilbert space H and $g\in C^1(H,\mathbf{R})$. Because the author assume that $g\in C^1(H,\mathbf{R})$ has a bounded and compact differential $dg(x)$, only if f is bounded his result can be used to discuss (1.1)–(1.2).

Compared with the above papers our theorem is a new result, which concerns multiple periodic solutions of (1.1)–(1.2) with resonance. In order to prove our theorem, we construct the corresponding functional:

$$\varphi(x)=\frac{1}{2}\int_0^1 \left|x'(t)\right|^2 dt-\int_0^1 F\big(t,x(t)\big)\,dt, \quad \forall x\in E, \tag{1.3}$$

where $F(t,u)=\int_0^u f(t,s)\,ds$, $u\in\mathbf{R}$, and E will be described in Section 2. This functional $\varphi(x)$ is continuous differentiable on E, and any critical point of φ corresponds to a solution of (1.1)–(1.2).

In Section 3, we will give the proof of Theorem 1.1 by the Morse theory following [13,10].

2. Index theory for linear Duffing equations

For any $a\in L^\infty[0,1]$, consider the following equations:

$$x''(t)+a(t)x=0, \tag{2.1}$$

$$x(0)-x(1)=x'(0)-x'(1)=0. \tag{2.2}$$

Define a Hilbert space $E:=\{x\in H^1[0,1] \mid x(0)=x(1)\}$ with norm $\|x\|_E:=\{\int_0^1[|x(t)|^2+|x'(t)|^2]\,dt\}^{\frac{1}{2}}$ and

$$q_a(x,y)=\int_0^1\big[x'(t)y'(t)-a(t)x(t)y(t)\big]\,dt, \quad \forall x,y\in E. \tag{2.3}$$

From [1,14], we have the following properties.

Proposition 2.1. *For any $a \in L^\infty[0,1]$:*

(1) *The E can be divided into three parts*:

$$E = E^+(a) \oplus E^0(a) \oplus E^-(a)$$

such that q_a is positive definite, null and negative definite on $E^+(a)$, $E^0(a)$ and $E^-(a)$ respectively. Furthermore, $E^0(A)$ and $E^-(A)$ are finitely dimensional. We call $\nu(a) := \dim E^0(a)$ and $i(a) := \dim E^+(a)$ the nullity and index respectively.

(2) $(i(a), \nu(a)) \in \mathbf{N} \times \{0,1,2\}$.

(3) *$\nu(a)$ is the dimension of the solution subspace of* (2.1)–(2.2)*, and $i(a) = \sum_{s<0} \nu(a+s)$.*

(4) *If $a_1 \leqslant a_2$, then $i(a_1) \leqslant i(a_2)$ and $i(a_1) + \nu(a_1) \leqslant i(a_2) + \nu(a_2)$; if $a_1 < a_2$, then $i(a_1) + \nu(a_1) \leqslant i(a_2)$.*

(5) *There exists $\delta > 0$ such that*

$$q_a(u,u) \geqslant \delta \|u\|_E^2, \quad \forall u \in E^+(a).$$

Remark. From (4), we can see that the index $i(b)$ is monotone with respect to B.

Example 2.2. Let $a(t) = 0$. Then (2.1) and (2.2) have nontrivial solutions $x = c\ (\in \mathbf{R}) \neq 0$. So $\nu(0) = 1$, $i(0) = 0$. If $a(t) = 4\pi^2$, then (2.1) and (2.2) have solutions $c_1 \cos 2\pi t$ and $c_2 \sin 2\pi t$, $c_1 \neq 0$, $c_2 \neq 0$. So $\nu(4\pi^2) = 2$, $i(4\pi^2) = 1$. If $a(t) = 4k^2\pi^2$, then (2.1) and (2.2) have solutions $c_{k1} \cos 2k\pi t$ and $c_{k2} \sin 2k\pi t$, $c_{k1} \neq 0$, $c_{k2} \neq 0$. So by (3) of Proposition 2.1, $i(4k^2\pi^2) = \sum_{s<0} \nu(4k^2\pi^2 + s) = \sum_{n=1}^{k-1} \nu(4n^2\pi^2) = 1 + 2(k-1) = 2k-1$, $\nu(4k^2\pi^2) = 2$.

The following lemmas are useful for us to prove the result.

Lemma 2.3. *The norm $\|x\|_C := \sup_{0 \leqslant t \leqslant 1} |x(t)| \leqslant C_* \|x\|_E$, for any $x \in E$, where $C_*\ (\in \mathbf{R})$ is a constant.*

Lemma 2.4. *If (H_2) holds, then we have that $E = \mathbf{R} \oplus E^+(\lambda)$.*

Proof. By (1) of Proposition 2.1 and conditions $\nu(\lambda) = 0$, $i(\lambda) = 1$, we have that

$$E^0(\lambda) = \{\theta\}, \qquad \dim E^-(\lambda) = 1. \tag{2.4}$$

By (1) of Proposition 2.1 and (2.4), we know that with respect to $\lambda \in L^\infty[0,1]$, the following decomposition holds:

$$\begin{aligned} E &= E^-(\lambda) \oplus E^0(\lambda) \oplus E^+(\lambda) \\ &= E^-(\lambda) \oplus E^+(\lambda). \end{aligned} \tag{2.5}$$

Since $E^-(\lambda)$ is one dimensional space, we can assume that $\{e^-\}$ is a base of $E^-(\lambda)$, i.e. $E^-(\lambda) = \mathrm{span}\{e^-\}$. So for any $x \in E$, we have

$$x = x^- + x^+ = c_0 e^- + x^+,$$

where $x^- \in E^-(\lambda)$, $x^+ \in E^+(\lambda)$, and c_0 is a constant. For $1\ (\in \mathbf{R}) \in E$, we have the decomposition $1 = c_1 e^- + e^+$, where $e^+ \in E^+(\lambda)$ and c_1 is a constant. It is obvious that $c_1 \neq 0$. Indeed, if $c_1 = 0$, then $1 = e^+$, we will have a contradiction that $q_\lambda(1,1) = -\int_0^1 \lambda(t) 1^2\, dt < 0$ and $q_\lambda(e^+, e^+) = \int_0^1 (e^{+\prime}, e^{+\prime}) - \lambda(t)(e^+, e^+)\, dt \geqslant 0$. So we obtain

$$\begin{aligned} x &= c_0 e^- + x^+ \\ &= \frac{c_0}{c_1}\bigl(c_1 e^- + e^+\bigr) - \frac{c_0}{c_1} e^+ + x^+ \\ &= \frac{c_0}{c_1} + \left(x^+ - \frac{c_0}{c_1} e^+\right). \end{aligned}$$

We have proved that $E = \mathbf{R} \cup E^+(\lambda)$. It is also obvious that $\mathbf{R} \cap E^+(\lambda) = \{\theta\}$. In fact that if $x\ (\neq \theta) \in \mathbf{R} \cap E^+(\lambda)$, we have that on the one hand for $x \in \mathbf{R}$, $q_\lambda(x,x) = -\int_0^1 \lambda(t) x^2\, dt < 0$; on the other hand for $x \in E^+(\lambda)$, $q_\lambda(x,x) = \int_0^1 |x'(t)|^2\, dt - \int_0^1 \lambda(t) x^2(t)\, dt \geqslant 0$. By (1) of Proposition 2.1 and (2.3), we have $x = \theta$. This is a contradiction. This completes the proof. □

In order to prove Theorem 1.1, we need some lemmas. Let X be a Hilbert space and $\psi \in C^1(X, \mathbf{R})$. As in [13], let $K = \{x \in X \mid \psi'(x) = \theta\}$, $\psi_m = \{x \in X \mid \psi(x) \leqslant m\}$. For an isolated critical point x_0, the critical group is defined by $C_q(\psi, x_0) =$

$H_q(\psi_c \cap U, (\psi_c \setminus \{x_0\}) \cap U; \mathbf{R})$ for $q = 0, 1, 2, 3, \ldots,$ where U is a neighborhood of $\{x_0\}$ such that $K \cap (\psi_c \cap U) = \{x_0\}$ and $c = \psi(x_0)$.

When $\psi \in C^2(X, \mathbf{R})$ and $p \in K$, we have that $\psi''(p)$ is a self-adjoint operator. We call the dimension of negative space corresponding to the spectral decomposing the Morse index of p and denote it by $m^-(\psi''(p))$, and denote $m^0(\psi''(p)) = \dim\ker\psi''(p)$. If $\psi''(p)$ has a bounded inverse we say that p is nondegenerate.

From [13, Chapter 3, Theorem 3.1, Chapter 2, Theorems 5.1, 5.2, Corollary 5.2], one can prove

Lemma 2.5. *Assume $\psi \in C^2(X, \mathbf{R})$ satisfies the* (PS) *condition, $\psi'(\theta) = \theta$, and there is a positive integer γ such that $\gamma \,\bar{\in}\, [m^-(\psi''(\theta)), m^-(\psi''(\theta)) + m^0(\psi''(\theta))]$ and $H_q(X, \psi_m; \mathbf{R}) = \delta_{q\gamma}\mathbf{R}$ for some regular $m < \psi(\theta)$, where $\delta_{q\gamma} = \begin{cases} 1, & q=\gamma, \\ 0, & q\neq\gamma. \end{cases}$ Then ψ has a critical point $p_0 \neq \theta$ with $C_\gamma(\psi, p_0) \neq 0$. Moreover, if θ is a nondegenerate critical point, and $m^0(\psi''(p_0)) \leqslant |\gamma - m^-(\psi''(\theta))|$, then ψ has another critical point $p_1 \neq p_0, \theta$.*

3. Proof of Theorem 1.1

The proof of Theorem 1.1 depends on the following lemma.

Lemma 3.1. *Under assumptions (H_1), (H_2), the functional φ satisfies the* (PS)*-condition.*

Proof. For $\{x_n\} \subset E$, $\varphi'(x_n) \to \theta$, and $\varphi(x_n)$ being bounded, we shall find a convergent subsequence in E. By (1.3), for $u \in E$, we have

$$\langle \varphi'(x_n), u\rangle = \int_0^1 \big(x_n'(t), u'(t)\big)\, dt - \int_0^1 \big(f\big(t, x_n(t)\big), u(t)\big)\, dt, \tag{3.1}$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbf{R}$. Next, we will prove $\{\|x_n\|_E\}_1^\infty$ is bounded. Indeed, it suffices to prove that $\|x_n\|_C$ is bounded. By a contradiction, we assume that $\|x_n\|_C \to +\infty$, as $n \to \infty$.

Defining

$$b_n(t) = \begin{cases} \frac{f(t, x_n(t))}{x_n(t)}, & |x_n(t)| \geqslant r, \\ \lambda(t), & |x_n(t)| < r, \end{cases} \quad \text{and} \quad h_n(t) = f\big(t, x_n(t)\big) - b_n(t)x_n(t), \tag{3.2}$$

from (H_2), and $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ being continuous, we have

$$0 \leqslant b_n(t) \leqslant \lambda(t) \quad \text{and} \quad \big|h_n(t)\big| < C_0, \tag{3.3}$$

where C_0 is a constant. Then we get

$$f\big(t, x(t)\big) = b_n(t)x_n(t) + h_n(t). \tag{3.4}$$

By (3.1), it follows that

$$\int_0^1 \big(x_n'(t), u'(t)\big)\, dt = \int_0^1 \big(f\big(t, x_n(t)\big), u(t)\big)\, dt + \langle \varphi'(x_n), u\rangle. \tag{3.5}$$

Assuming $y_n = \frac{x_n}{\|x_n\|_C}$, by (3.4), and multiplying $\|x_n\|_C^{-1}$ on both sides of (3.5), we can get

$$\int_0^1 \big(y_n'(t), u'(t)\big)\, dt = \int_0^1 b_n(t) y_n u\, dt + \|x_n\|_C^{-1}\left(\int_0^1 h_n(t)u(t)\, dt + \varphi'(x_n)u\right). \tag{3.6}$$

Furthermore, we add $\int_0^1 (y_n(t), u(t))\, dt$ on two sides of (3.6) to obtain

$$\begin{aligned}(y_n, u)_E &= \int_0^1 y'u'dt + \int_0^1 y_n u\, dt \\ &= \int_0^1 b_n(t)y_n u\, dt + \int_0^1 y_n u\, dt + \|x_n\|_C^{-1}\left(\int_0^1 h_n(t)u(t)\, dt + \varphi'(x_n)u(t)\right).\end{aligned}$$

So, by $\|y_n\|_{L^2} = (\int_0^1 y_n^2(t)\,dt)^{\frac{1}{2}} \leqslant \|y_n\|_C = 1$ and (3.3) we have

$$\begin{aligned}\|y_n\|_E = \sup_{\|u\|_E\leqslant 1}(y_n,u)_E &\leqslant \int_0^1 \big(b_n(t)y_n(t)u\big)\,dt + \int_0^1 \big(y_n(t)u\big)\,dt + C_2\\ &\leqslant C_3\|y_n\|_{L^2}\|u\|_{L^2} + C_2\\ &\leqslant C^*,\end{aligned}$$

where C_2, C_3 and C^* are constants. So $\{\|y_n\|_E\}_1^\infty$ is bounded. Then $\{y_n\}$ has a convergent subsequence. Without loss of generality, we also denote it by $\{y_n\}$. Then $y_n \rightharpoonup y_0$ in E and $y_n \to y_0$ in $C[0,1]$. By inequality $0 \leqslant b_n(t) \leqslant \lambda(t)$, we have $b_n \rightharpoonup b_0$ in $L^2[0,1]$. Then taking the limits on both sides of (3.6), we have $\int_0^1 y_0' u'\,dt = \int_0^1 b_0(t)y_0(t)u(t)\,dt$, for any $u \in E$, i.e.

$$\int_0^1 y_0' u'\,dt - \int_0^1 b_0(t)y_0 u\,dt = 0 \quad \forall u \in E. \tag{3.7}$$

From (3.7) and [1], we have that y_0 is a solution of the following problem:

$$\begin{aligned}&y''(t) + b_0(t)y = 0 \quad \text{a.e. } t \in [0,1],\\ &y(0) - y(1) = 0 = y'(0) - y'(1).\end{aligned} \tag{3.8}$$

What's more, because of $0 \leqslant b_n(t) \leqslant \lambda(t)$, we have $b_0 = 0$. In fact, by the meaning of the notation "$<$" and "$\leqslant$", on the one hand, if $b_0(t) = \lambda(t)$, then $\nu(b_0) = \nu(\lambda) = 0$. Therefore, by the definition $\nu(\cdot)$, this means that (3.8) only has a trivial solution. In fact, by $\|y_n\|_C = 1$, we obtain $\|y_0\|_C = 1$. So (3.8) has a nontrivial solution. This is a contradiction. On the other hand, if $0 < b_0(t) < \lambda(t)$, then $1 = i(0) + \nu(0) \leqslant i(b_0) \leqslant i(\lambda) = 1$ holds. While y_0 is a nontrivial solution of (3.8), this leads to $\nu(b_0) = 1$. So by (4) of Proposition 2.1, we get $2 = i(b_0) + \nu(b_0) \leqslant i(\lambda) = 1$. This is also a contradiction. All in all, from discussion above, we obtain the conclusion that $b_0 = 0$. So we immediately get $y_0 = \text{constant}\ (\neq 0)$, which is the solution of (3.8).

Since y_0 is a nonzero constant, there are two cases about y_0. One is that $y_0 > 0$, the other is that $y_0 < 0$. Firstly, we discuss the situation that $y_0 > 0$. If $y_0 > 0$, for $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ such that for $n > N$, $|y_n - y_0| < \varepsilon$ holds. Here, we take the ε such that $y_0 - \varepsilon > 0$, i.e. when $n > N$, $y_n(t)$ belong to the neighborhood of y_0, $(y_0 - \varepsilon, y_0 + \varepsilon)$, for all $t \in [0,1]$. This means that $y_n(t) > y_0 - \varepsilon$, as $n > N$, for all $t \in [0,1]$.

So by $y_n = \frac{x_n(t)}{\|x_n\|_C}$, we can get that for any $t \in [0,1]$, $x_n(t) \geqslant (y_0 - \varepsilon)\|x_n\|_C > 0$ if $n > N$. Then $x_n(t) \to +\infty$ for all $t \in [0,1]$, as $\|x_n\|_C \to \infty$. By the assumption that $\|x_n\|_C \to \infty$, as $n \to \infty$, taking the limits on both sides of (3.5) and letting $u = 1$, we can obtain

$$\int_0^1 f\big(t, x_n(t)\big)\,dt \to 0, \quad \text{as } n \to \infty. \tag{3.9}$$

By Fatou's Lemma and (3.9), we have

$$\begin{aligned}\varliminf_{n\to\infty} \int_0^1 f\big(t,x_n(t)\big)\,dt = 0 &\geqslant \int_0^1 \varliminf_{n\to\infty} f\big(t,x_n(t)\big)\,dt\\ &\geqslant \int_0^1 f(t,+\infty)\,dt,\end{aligned}$$

a contradiction to assumption (H_1). Hence, if $y_0 > 0$, this has led to a contradiction. Secondly, in a similar way, we can show that the case $y_0 < 0$ will also bring a contradiction. Therefore, the sequence $\{\|x_n\|_C\}_1^\infty$ is a bounded sequence. By the equality $x_n = y_n\|x_n\|_C$ and the fact that $\|y_n\|_E$ is bounded, we can get that $\{\|x_n\|_E\}_1^\infty$ is bounded in E. Furthermore, $\{x_n\}_1^\infty$ has a weak convergent subsequence in E, without loss of generality, still denoted by $\{x_n\}_1^\infty$. So we have $x_n \rightharpoonup x_0$ in E and $x_n \to x_0$ in $C[0,1]$. In addition, by (3.1), we also have

$$\int_0^1 \big(x_0'(t), u'(t)\big)\,dt - \int_0^1 \big(f\big(t,x_0(t)\big), u(t)\big)\,dt = 0. \tag{3.10}$$

At last, we only need to finish the mission that $x_n \to x_0$ in E. Indeed, by (3.5), (3.10) and $x_n \rightharpoonup x_0$, we obtain the fact that

$$\|x_n - x_0\|_E = \sup_{\|u\|_E \leqslant 1} (x_n - x_0, u)_E = \sup_{\|u\|_E \leqslant 1} \left[\int_0^1 \left(x_n' - x_0', u'\right) dt + \int_0^1 (x_n - x_0, u)\, dt \right]$$
$$= \sup_{\|u\|_E \leqslant 1} \left\{ \int_0^1 \left[f\left(t, x_n(t)\right) - f\left(t, x_0(t), u(t)\right)\right] dt + \varphi'(x_n)u + \int_0^1 (x_n - x_0, u)\, dt \right\} \to 0, \quad \text{as } n \to \infty.$$

The (PS) condition is verified. □

After giving the preliminary work, we can prove Theorem 1.1. The following proof comes from [13, Theorem 3.3], [14, Theorem 2.2.4].

Proof of Theorem 1.1. Since $\nu(a) \leqslant 2$ for any $a \in L^\infty[0,1]$, by Lemma 2.5, we only need to prove

$$H_q(E, \varphi_z; \mathbf{R}) \cong \delta_{q\gamma} \mathbf{R} \tag{3.11}$$

for $-z > -\varphi(\theta)$ large enough, where $\gamma = \nu(0) = i(\lambda) = 1$. By Lemma 2.4, we know that E can be split into two subspaces $\mathbf{R}$ and $E^+(\lambda)$, i.e.

$$E = \mathbf{R} \oplus E^+(\lambda).$$

Next, we will take two steps to obtain the proof of (3.11).

First step: For $-z > -\varphi(\theta)$ large enough, we have

$$H_q(E, \varphi_z; \mathbf{R}) \cong H_q(\mathcal{M}, \mathcal{M} \cap \varphi_z; \mathbf{R}), \quad q = 0, 1, 2, \ldots, \tag{3.12}$$

where $\mathcal{M} \subset H$ will be defined later. By assumption, for any $y \in E$, we have

$$\left\langle \varphi'(x), y \right\rangle = \int_0^1 x'(t) y'(t)\, dt - \int_0^1 f\left(t, x(t)\right) y(t)\, dt. \tag{3.13}$$

We will consider the behavior of f in two subintervals of $[0,1]$. One is $\{t \mid x(t) \geqslant r\}$, the other is $\{t \mid |x(t)| < r\}$. Since f is continuous on $[0,1] \times (-\infty, +\infty)$, it is obvious that $f(t, x(t))$ is bounded on $\{t \mid |x(t)| < r\}$. So there exists a constant $M_1 \in \mathbf{R}$ such that $|f(t, x(t))| < M_1$ when $|x(t)| < r$.

By Lemma 2.4, we have a decomposition with respect to $x(t) \in E$, i.e. there exist $x_+(t) \in E^+(\lambda)$, $c \in \mathbf{R}$ such that $x(t) = x_+(t) + c$. When $|x_+(t) + c| < r$, we have $|c| < r + |x_+(t)| < r + \|x_+\|_C$. Furthermore, we get

$$\left| c \int_{|x_+(t)+c|<r} f\left(t, x_+(t) + c\right) dt \right| < M_1\left(r + \|x_+\|_C\right). \tag{3.14}$$

By (H_1), we have

$$\begin{aligned}
\int_{|x_+ + c| \geqslant r} f(t, x_+ + c)(x_+ - c)\, dt &= \int_{|x_+ + c| \geqslant r} \frac{f(t, x_+ + c)}{x_+ + c} (x_+ + c)(x_+ - c)\, dt \\
&= \int_{|x_+ + c| \geqslant r} \frac{f(t, x_+ + c)}{x_+ + c} \left(x_+^2 - c^2\right) dt \\
&= \int_{|x_+ + c| \geqslant r} \frac{f(t, x_+ + c)}{x_+ + c} x_+^2\, dt - \int_{|x_+ + c| \geqslant r} c^2 \frac{f(t, x_+ + c)}{x_+ + c}\, dt \\
&\leqslant \int_{|x_+ + c| \geqslant r} \lambda(t) \left|x_+(t)\right|^2 dt \leqslant \int_0^1 \lambda(t) \left|x_+(t)\right|^2 dt,
\end{aligned} \tag{3.15}$$

and by (3.14), we have

$$\begin{aligned}
\int_{|x_+ + c| < r} f(t, x_+ + c)(x_+ - c)\, dt &= \int_{|x_+ + c| < r} f(t, x_+ + c) x_+\, dt - c \int_{|x_+ + c| < r} f(t, x_+ + c)\, dt \\
&\leqslant M_1 \|x_+\|_C + M_1\left(r + \|x_+\|_C\right).
\end{aligned} \tag{3.16}$$

By (3.13), (3.15), (3.16), (5) of Proposition 2.1 and Lemma 2.3, we obtain

$$\begin{aligned}
\langle \varphi'(x), x_+ - c\rangle &= \int_0^1 \bigl(x'_+(t) + c', x'_+(t) - c'\bigr)\,dt - \int_0^1 f(t, x_+ + c)(x_+ - c)\,dt \\
&= \int_0^1 \bigl|x'_+(t)\bigr|^2\,dt - \Bigl(\int_{|x_+ + c|\geqslant r} f(t, x_+ + c)(x_+ - c)\,dt + \int_{|x_+ + c|< r} f(t, x_+ + c)(x_+ - c)\,dt\Bigr) \\
&\geqslant \int_0^1 \bigl|x'_+(t)\bigr|^2\,dt - \int_0^1 \lambda(t)x_+^2\,dt - M_1\|x_+\|_C - M_1\bigl(r + \|x_+\|_C\bigr) \\
&= q_\lambda(x_+, x_+) - 2M_1\|x_+\|_C - rM_1 \\
&\geqslant C_1\|x_+\|_E^2 - C_4\|x_+\|_E - rM_1,
\end{aligned}$$

where $C_1 > 0$, $C_4 > 0$ are constants. And hence, there exists $R_0 > 0$ such that

$$\langle \varphi'(x), x_+ - c\rangle > 1, \quad \forall x \in E \text{ with } \|x_+\|_E > R_0.$$

Set $\mathcal{M} = (E^+(\lambda) \cap B_{R_0}) \oplus \mathbf{R}$, where $B_{R_0} = \{x \in E \mid \|x\|_E \leqslant R_0\}$. We want to define a deformation from (E, φ_z) to $(\mathcal{M}, \mathcal{M} \cap \varphi_z)$. Since for every $x = x_+ + c \in \mathcal{M}$, f is decreasing along vector field $V(x) = -x_+ + c$, we can define the flow $\sigma = \sigma(t, x) = e^{-t}x_+ + e^t c$ and $T_x = \ln\|x_+\|_E - \ln R_0$, which is the first time that $\sigma(t, x)$ arrives at $\mathcal{M}$. Then the deformation is

$$\eta(t, x_+ + c) = \begin{cases} x_+ + c, & \|x_+\|_E \leqslant R_0, \\ \sigma(T_x t, x), & \|x_+\|_E > R_0. \end{cases}$$

One can verify that $\eta : [0,1] \times E \to E$ is continuous and satisfies

$$\begin{aligned}
&\eta(0,\cdot) = id_E, \qquad \eta(1, E) \subset \mathcal{M}, \qquad \eta(1, \varphi_z) \subset \mathcal{M} \cap \varphi_z, \\
&\eta(t, \varphi_z) \subset \varphi_z, \qquad \eta(t,\cdot)|_{\mathcal{M}} = id_{\mathcal{M}} \quad \forall t \in [0,1].
\end{aligned}$$

Then, $(\mathcal{M}, \mathcal{M} \cap \varphi_z)$ is a deformation retract of (E, φ_z). So (3.12) is verified.

Second step: We begin to prove that for any $-z > -\varphi(\theta)$ large enough, we have

$$H_q(\mathcal{M}, \mathcal{M} \cap \varphi_z; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R}. \tag{3.17}$$

In fact, assuming that $|x(t)| = |x_+ + c| \geqslant r$, we will face two cases: one is $x_+ + c \geqslant r$, another is $x_+ + c \leqslant -r$. Firstly, we analyze the situation that $x_+ + c \geqslant r$. By assumption (H_2), we have

$$\begin{aligned}
F\bigl(t, x(t)\bigr) &= \int_0^{x(t)} f(t, s)\,ds \leqslant \int_0^r f(t, s)\,ds + \int_r^{x(t)} \lambda(t)s\,ds \\
&\leqslant M_2 + \frac{\lambda(t)}{2}\bigl(x^2(t) - r^2\bigr) \\
&= M_2 + \frac{\lambda(t)}{2}x_+^2 + c\lambda(t)x_+ + \frac{\lambda(t)}{2}c^2 - \frac{\lambda(t)}{2}r^2 \\
&= c^2\frac{\lambda(t)}{2} + c\lambda(t)x_+ + \frac{\lambda(t)}{2}x_+^2 - \frac{\lambda(t)}{2}r^2 + M_2,
\end{aligned} \tag{3.18}$$

where $M_2 > 0$ is a constant. Furthermore, by (3.18), we can get

$$\begin{aligned}
\varphi(x) &= \frac{1}{2}\int_0^1 \bigl|x'(t)\bigr|^2\,dt - \int_0^1 F\bigl(t, x(t)\bigr)\,dt \\
&\geqslant \frac{1}{2}\int_0^1 x_+'^2\,dt - \int_0^1 \frac{\lambda(t)}{2}\,dt\cdot c^2 - \int_0^1 \lambda(t)x_+\,dt\cdot c - \frac{1}{2}\int_0^1 \frac{\lambda(t)}{2}x_+^2\,dt + r^2\int_0^1 \frac{\lambda(t)}{2}x_+\,dt - M_2.
\end{aligned} \tag{3.19}$$

Secondly, when $x_+ + c \leqslant -r$, we have

$$F\big(t,x(t)\big)=\int_0^{x(t)} f(t,s)\,ds \leqslant \int_0^{-r} f(t,s)\,ds+\int_{-r}^{x(t)} \lambda(t)s\,ds$$
$$\leqslant M_3+\frac{\lambda(t)}{2}\big(x^2(t)-r^2\big)$$
$$=c^2\frac{\lambda(t)}{2}+c\lambda(t)x_+ +\frac{\lambda(t)}{2}x_+^2-\frac{\lambda(t)}{2}r^2+M_3,$$

where $M_3>0$ is a constant. Then we also get

$$\varphi(x)=\frac{1}{2}\int_0^1 \big|x'(t)\big|^2\,dt-\int_0^1 F\big(t,x(t)\big)\,dt$$
$$\geqslant \frac{1}{2}\int_0^1 x_+'^2\,dt-\int_0^1 \frac{\lambda(t)}{2}\,dt\cdot c^2-\int_0^1 \lambda(t)x_+\,dt\cdot c-\frac{1}{2}\int_0^1 \frac{\lambda(t)}{2}x_+^2\,dt+r^2\int_0^1 \frac{\lambda(t)}{2}x_+\,dt-M_3. \tag{3.20}$$

Meanwhile, by (H_2), we know that $f(t,x)\geqslant 0$ as $x\geqslant r$ and $f(t,x)\leqslant 0$ as $x\leqslant -r$. Firstly, we analyze the situation that $f(t,x)\geqslant 0$ as $x\geqslant r$. Since $f(t,+\infty)=\liminf_{x\to+\infty} f(t,x)=\liminf_{x\to+\infty,y\geqslant x} f(t,y)$ and $\inf_{y\geqslant x} f(t,y)$ is a monotonically increasing nonnegative function with respect to $x\geqslant r>0$, by (H_1), we have $\int_0^1 f(t,\infty)\,dt=\int_0^1 \liminf_{x\to+\infty,y\geqslant x} f(t,y)\,dt=\lim_{x\to+\infty}\int_0^1 \inf_{y\geqslant x} f(t,y)\,dt>0$. Then $\exists x^*\in\mathbf{R}$, for all $l>x^*$, $\int_0^1 \inf_{y\geqslant l} f(t,y)\,dt>0$ holds. So letting $x=x_++c>l$, where l is fixed and $l>x^*$, we have

$$F(t,x_++c)=\int_0^{x_++c} f(t,s)\,ds$$
$$=\int_0^{l} f(t,s)\,ds+\int_l^{x_++c} f(t,s)\,ds$$
$$\geqslant M_4+(x_++c-l)\inf_{x_++c\geqslant y\geqslant l} f(t,y). \tag{3.21}$$

Furthermore, by (3.21), we obtain

$$\varphi(x)=\frac{1}{2}\int_0^1 \big|x'(t)\big|^2\,dt-\int_0^1 F\big(t,x(t)\big)\,dt$$
$$\leqslant \frac{1}{2}\int_0^1 \big|x_+'(t)\big|^2\,dt-M_4-(x_++c-l)\int_0^1 \inf_{x_++c\geqslant y\geqslant l} f(t,y)\,dt.$$

So we get $\varphi(x)\to-\infty$, as $c\to+\infty$, uniformly in $x_+\in E^+(\lambda)\cap B_{R_0}$. Secondly, in a similar way, we also get $\varphi(x)\to-\infty$, as $c\to-\infty$, uniformly in $x_+\in E^+(\lambda)\cap B_{R_0}$.

We obtain from (3.19), (3.20), and the analysis above that

$$\varphi(x)\to-\infty \quad\Leftrightarrow\quad |c|\to+\infty \quad \text{uniformly in } x_+\in E^+(\lambda)\cap B_{R_0}.$$

Thus, there exist $T>0$, $z_1<z_2<-T$, $R_1>R_2>R_0$ such that

$$\big(E^+(\lambda)\cap B_{R_0}\big)\oplus\big(\mathbf{R}\setminus[-R_1,R_1]\big)\subset\varphi_{z_1}\cap\mathcal{M}\subset\big(E^+(\lambda)\cap B_{R_0}\big)\oplus\big(\mathbf{R}\setminus[-R_2,R_2]\big)\subset\varphi_{z_2}\cap\mathcal{M}. \tag{3.22}$$

For the sake of convenience, we set $\mathcal{N}_R=(E^+(\lambda)\cap B_{R_0})\oplus(\mathbf{R}\setminus[-R,R])$. Then (3.22) can also be denoted as

$$\mathcal{N}_{R_1}\subset\varphi_{z_1}\cap\mathcal{M}\subset\mathcal{N}_{R_2}\subset\varphi_{z_2}\cap\mathcal{M}.$$

We now begin to define a deformation from $\mathcal{M}\cap\varphi_{z_2}$ to $\mathcal{M}\cap\varphi_{z_1}$. For every $x\in\mathcal{M}\cap(\varphi_{z_2}\setminus\varphi_{z_1})$, since the flow is defined by $\sigma(t,x)=e^{-t}x_++e^tc$, $\varphi(\sigma(t,x))$ is continuous with respect to t, $\varphi(\sigma(0,x))=\varphi(x)>z_1$ and $\varphi(\sigma(t,x))\to-\infty$ as $t\to+\infty$, so the time $t=T_1(x)$ arriving at $\varphi_{z_1}\cap\mathcal{M}$ exists uniquely and is defined by $\varphi(\sigma(t,x))=z_1$. Since

$$\frac{d\varphi(\sigma(t,x))}{dt} = \langle \varphi'(\sigma(t,x)), \sigma'(t,x)\rangle$$
$$= \langle \varphi'(e^{-t}x_+ + e^t c), -e^{-t}x_+ + e^t c\rangle \leqslant -1$$

as $t > 0$, the continuity of $t = T_1(x)$ comes from the implicit function theorem. Define

$$\eta_1(t,x) = x, \quad \forall x \in \varphi_{z_1} \cap \mathcal{M}$$
$$= \sigma(T_1(x)t, x), \quad \forall x \in \mathcal{M} \cap (\varphi_{z_2} \setminus \varphi_{z_1}),$$

then $\eta_1 : [0,1] \times \varphi_{z_2} \cap \mathcal{M} \to \varphi_{z_2} \cap \mathcal{M}$ is continuous, and is a deformation from $\varphi_{z_2} \cap \mathcal{M}$ to $\varphi_{z_1} \cap \mathcal{M}$ and $\tau_1 = \eta(1,(\cdot)) : \varphi_{z_2} \cap \mathcal{M} \to \varphi_{z_1} \cap \mathcal{M}$ is a strong deformation retract. Hence,

$$H_q(\varphi_{z_2} \cap \mathcal{M}, \varphi_{z_1} \cap \mathcal{M}; \mathbf{R}) \cong 0. \tag{3.23}$$

Recall that for any topological spaces $Z \subseteq Y \subseteq X$, we have exact sequences

$$H_q(Y, Z; \mathbf{R}) \to H_q(X, Z; \mathbf{R}) \to H_q(X, Y; \mathbf{R}) \to H_{q-1}(Y, Z; \mathbf{R}).$$

From (3.22), in order to prove

$$H_q(\mathcal{M}, \varphi_{z_2} \cap \mathcal{M}; \mathbf{R}) \cong H_q(\mathcal{M}, \mathcal{N}_{R_1}; \mathbf{R}) \tag{3.24}$$

we only need to prove

$$H_q(\varphi_{z_2} \cap \mathcal{M}, \mathcal{N}_{R_1}; \mathbf{R}) \cong 0.$$

And from (3.23), it suffices to verify

$$H_q(\varphi_{z_1} \cap \mathcal{M}, \mathcal{N}_{R_1}; \mathbf{R}) \cong 0. \tag{3.25}$$

Let $\tau_2 : [0,1] \times \mathcal{N}_{R_2} \to \mathcal{N}_{R_2}$ satisfy

$$\tau_2(t, x_+ + c) = x_+ + c, \quad |c| > R_1$$
$$= x_+ + \frac{c}{|c|}(tR_1 + (1-t)c), \quad R_2 < |c| \leqslant R_1.$$

We can verify that $\tau := \tau_1 \circ \tau_2 : [0,1] \times \varphi_{z_1} \cap \mathcal{M} \to \varphi_{z_1} \cap \mathcal{M}$ is continuous and satisfies

$$\tau[0,x] = \eta_1(1, \tau_2(0,x)) = \eta_1(1, x_+ + c) = x_+ + c = x$$

for any $x \in \varphi_{z_1} \cap \mathcal{M}$. So $\tau[0,\cdot] = id_{\varphi_{z_1} \cap \mathcal{M}}$. And

$$\tau[t, x_+ + c] = \tau_1 \circ \tau_2[t,x] = \eta_1(1,x) = x,$$

for any $x \in \mathcal{N}_{R_1}$. So $\tau[t,\cdot]|_{\mathcal{N}_{R_1}} = id_{\mathcal{N}_{R_1}}$. We can also see that τ satisfies $\tau(1, \varphi_{z_1} \cap \mathcal{M}) \subset \mathcal{N}_{R_1}$, $\tau(1, \mathcal{N}_{R_1}) \subset \mathcal{N}_{R_1}$, $\tau(t, \mathcal{N}_{R_1}) \subset \mathcal{N}_{R_1}$. Then $(\mathcal{N}_{R_1}, \mathcal{N}_{R_1})$ is a deformation retract of $(\varphi_{z_1} \cap \mathcal{M}, \mathcal{N}_{R_1})$. This means (3.24) and hence (3.23) holds. Finally from (3.23) we have

$$H_q(\mathcal{M}, \mathcal{M} \cap \varphi_{z_2}; \mathbf{R}) \cong H_q\big((E^+(\lambda) \cap B_{R_0}) \oplus \mathbf{R}, (E^+(\lambda) \cap B_{R_0}) \oplus (\mathbf{R} \setminus [-R_1, R_1]); \mathbf{R}\big)$$
$$\cong H_q(\mathbf{R} \cap [-R_1, R_1]; \partial(\mathbf{R} \cap [-R_1, R_1]); \mathbf{R})$$
$$= \delta_{q\gamma}\mathbf{R}, \quad q = 0, 1, 2, 3, \ldots.$$

Here in the second $\cong$ we used the deformation $\zeta : [0,1] \times \mathcal{M} \to \mathcal{M}$ defined by $\zeta(t,x) = tx_+ + c$, and excision property. This is (3.17). And from (3.12) and (3.17), we can get (3.11). This completes the proof. $\square$

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